

UNIT ONE

Classical Mechanics

II

Dynamics of Systems of Particle

by Merkebu. B
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Center of Mass

- Consider a system composed of n particles, with each particle's mass described by m_α , where α is an index from $\alpha=1$ to $\alpha=n$. The total mass of the system is denoted by M .

$$M = \sum_{\alpha} m_{\alpha}, \text{ where the summation over } \alpha \text{ runs from } \alpha=1 \text{ to } \alpha=n.$$

- ① If the vector connecting to the origin with the α th particle is $\vec{r}_{\alpha}$, then the vector defining the position of the system's center of mass is relative to some origin O is

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^n m_{\alpha} \vec{r}_{\alpha} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \dots + m_n \vec{r}_n}{M} \quad \text{--- ②}$$

where $M = \sum_{\alpha} m_{\alpha}$

- * The center of mass can be noticed to be vector equation the center of mass position is vector $\vec{R}$ with three components (x, y, z) and the above equation is equivalent to the three equations.

$$X = \frac{1}{M} \sum_{\alpha=1}^n m_{\alpha} x_{\alpha}, \quad Y = \frac{1}{M} \sum_{\alpha=1}^n m_{\alpha} y_{\alpha}, \quad Z = \frac{1}{M} \sum_{\alpha=1}^n m_{\alpha} z_{\alpha}$$

and $\vec{R}$ can be expressed as $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

these are all about description of center of mass.

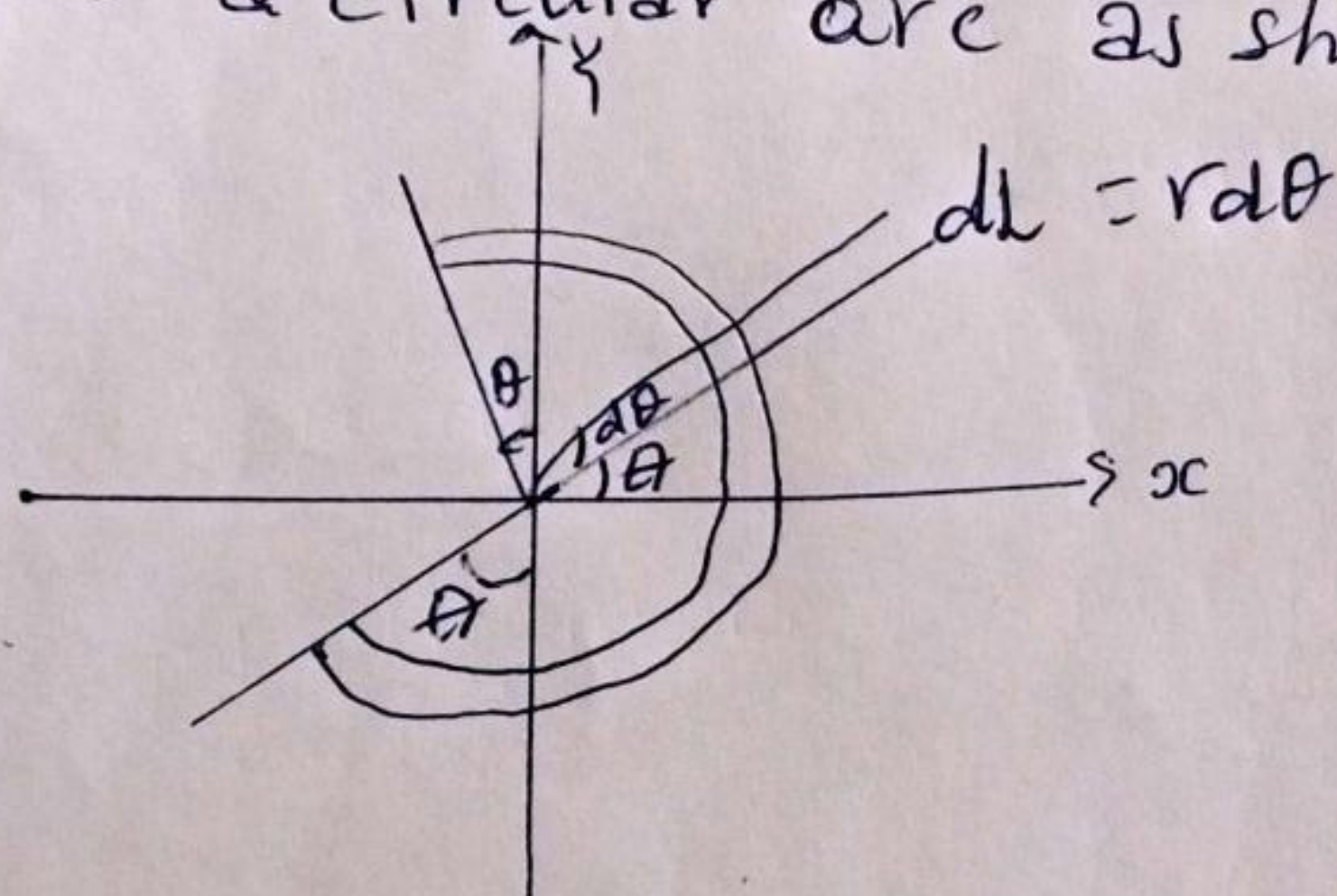
* For Continuous distribution of mass the position of the C.M. is defined by

$$R_{cm} = \frac{\int r dm}{\int dm} = \frac{\int r dm}{M}, \text{ where } M = \int dm$$

Thus $R_{cm} = \frac{1}{M} \int r dv$, where $dm = \rho dv$

ρ is the mass density
 dv is elements of volume.

Ex: Locate the center of mass of the homogeneous rod bent into shape of a circular arc as shown in figure below.



$$x = r \cos \theta$$

$$y = r \sin \theta$$

By symmetry $x_{cm} = 0$

$$x_{cm} = \frac{\int x dm}{\int dm} = \frac{\int x dl}{\int dl}$$

$$= \frac{\int r \cos \theta \lambda dl}{\int \lambda r d\theta}$$

$$= \frac{\int r \cos \theta \lambda r d\theta}{\int \lambda r d\theta}$$

$$= \frac{r \int \cos \theta d\theta}{\int d\theta}$$

$$dm = \lambda dl$$

$$dl = r d\theta$$

$$dm = \lambda r d\theta$$

$$= \frac{r \int_{-\pi/2}^{\pi/2} \cos \theta d\theta}{\int_{-\pi/2}^{\pi/2} d\theta}$$

(2)

$$x_{cm} = r \int \sin \theta \, d\theta \bigg|_{\frac{3\pi}{2} - \theta}^{\frac{\pi}{2} + \theta}$$

by Perkovu. b

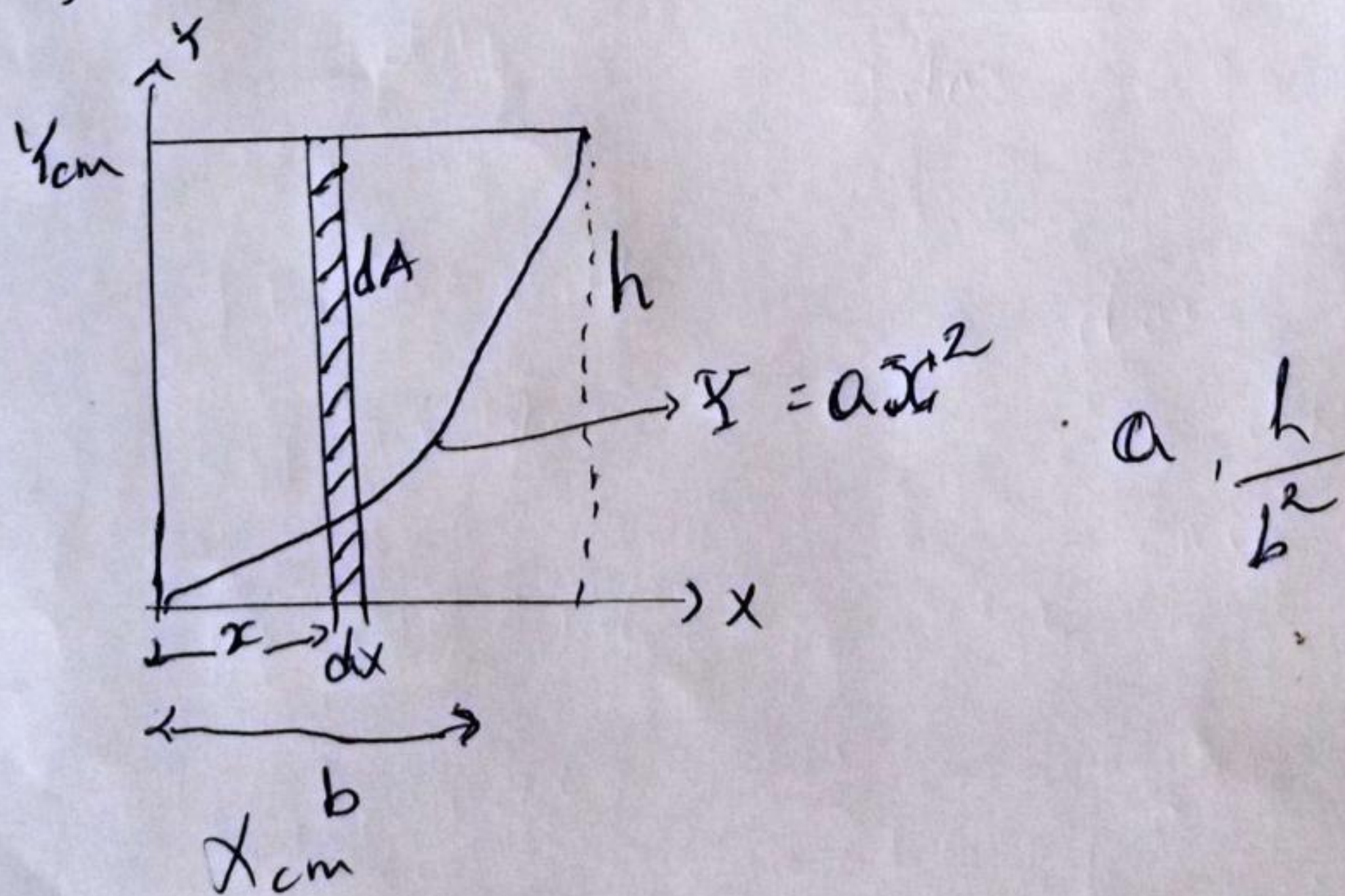
$$x_{cm} = r \left(\sin\left(\frac{\pi}{2} + \theta\right) - \sin\left(\frac{3\pi}{2} - \theta\right) \right) \bigg|_{\frac{3\pi}{2} - \theta}^{\frac{\pi}{2} + \theta}$$

$$x_{cm} = r \left(\sin\frac{\pi}{2} \cos\theta + \cos\frac{\pi}{2} \sin\theta - \left(\sin\frac{3\pi}{2} \cos\theta - \cos\frac{3\pi}{2} \sin\theta \right) \right) \bigg|_{-\pi + 2\theta}$$

$$x_{cm} = \frac{r(\cos\theta + \cos\theta)}{-\pi + 2\theta} = \frac{r(2\cos\theta)}{-\pi + 2\theta}$$

$$x_{cm} = \frac{r(2\cos\theta)}{-\pi + 2\theta}$$

Ex. 8.2. Locate the center of mass of the parabolic area as shown in figure below.



Solution

by Merhaba. 6

$$dA = x dx = y dx$$

$$dm = \delta dA$$

$$\int dA = \int x dx \quad - \text{integrating both sides}$$

$$A = \int x dx = \int_0^b ax^2 dx = \frac{1}{3} ax^3 \Big|_0^b \\ = \frac{1}{3} a (b^3 - 0^3) = \frac{1}{3} ab^3$$

$$A = \frac{1}{3} \frac{h}{b^2} b^3 = \frac{1}{3} hb$$

$$\boxed{A = \frac{1}{3} hb}, \quad \delta = \frac{M}{\frac{1}{3} hb}$$

$$x_{cm} = \frac{\int x dm}{M} = \frac{\int x \delta x dx}{M} = \frac{\delta}{M} \int x^2 dx = \frac{\delta}{M} \int a^2 x^4 dx$$

$$x_{cm} = \frac{\delta a^2}{M} \int_0^b x^4 dx = \frac{1}{5} \frac{\delta a^2}{M} x^5 \Big|_0^b \dots \dots \text{solve this :}$$

$$\boxed{x_{cm} = \frac{3}{5} h}$$

$x_{cm} = ?$ solve for this?

Home work

- ① Find the center of mass of a hemispherical shell of constant density and inner radius r_1 and outer radius r_2 .
- ② Find the center of mass of a uniform solid cone of base diameter $2a$ and height h .
- ③ Find the center of mass of a uniform solid cone of base diameter $2a$ and height h and a solid hemisphere of radius a where the two bases are touching.

④

Conservation of Linear Momentum

by Merkobu. B

- If a certain group of particles constitutes a system, then the resultant force acting on a particle within the system.
- Let say the α particle is in general composed of two parts. One part is the resultant of all force whose origin lies outside of the system is called the external force: F_{α}^e . The other part is the resultant of the force arising from the interaction of all the other $n-1$ particles with the α th particle. This is called internal force, f_{α} . Force f_{α} is given by the vector sum of all the individual force $f_{\alpha\beta}$.

$$f_{\alpha} = \sum_{\beta} f_{\alpha\beta}$$

where $f_{\alpha\beta}$ represents the force on the α th particle due to the β -particle

- The total force acting on the α th particle is therefore

$$F_{\alpha} = F_{\alpha}^{(e)} + f_{\alpha}$$

Also, according to the Newton's Third Law, we have

$$f_{\alpha\beta} = -f_{\beta\alpha}$$

Newton's Second Law for the α th particle can be written as

$$\dot{P}_{\alpha} = m_{\alpha} \ddot{r}_{\alpha} = F_{\alpha}^{(e)} + f_{\alpha} \text{ or}$$

$$\frac{d^2}{dt^2} (m_{\alpha} r_{\alpha}) = F_{\alpha}^{(e)} + \sum_{\beta} f_{\alpha\beta}$$

Summing this expression over α , we have

$$\frac{d^2}{dt^2} \sum_{\alpha} m_{\alpha} r_{\alpha} = \sum_{\alpha} F_{\alpha}^{(e)} + \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} f_{\alpha\beta}$$

(5)

$$\frac{d^2}{dt^2} MR = F^e + \sum_{\alpha \neq B} (f_{\alpha B} + f_{B\alpha})$$

by Newton's 3rd

thus the result is

$$M\ddot{R} = F^e$$

* The CM of a system moves as if it were a single particle of mass equal to the total mass of the system, acted on by the total external force, and independent of the nature of internal force (as long as they are internal).

$$f_{\alpha\beta} = -f_{\beta\alpha} \text{ (the weak form of Newton's third law)}$$

- the total linear momentum of the system

$$P = \sum_{\alpha} m_{\alpha} \ddot{r}_{\alpha} = \frac{d}{dt} \sum_{\alpha} m_{\alpha} \dot{r}_{\alpha} = \frac{d}{dt} (MR)$$

$$P = M\dot{R} \text{ and } \dot{P} = M\ddot{R} = F^e$$

thus the total linear momentum of the system is conserved if there is no external force.

- The linear momentum of the system is the same as if a single particle of mass M were located at the position of the CM and moving in the manner of the CM moves.

- The linear momentum of the system of external forces is constant and equal to the linear momentum of center of mass (CM).

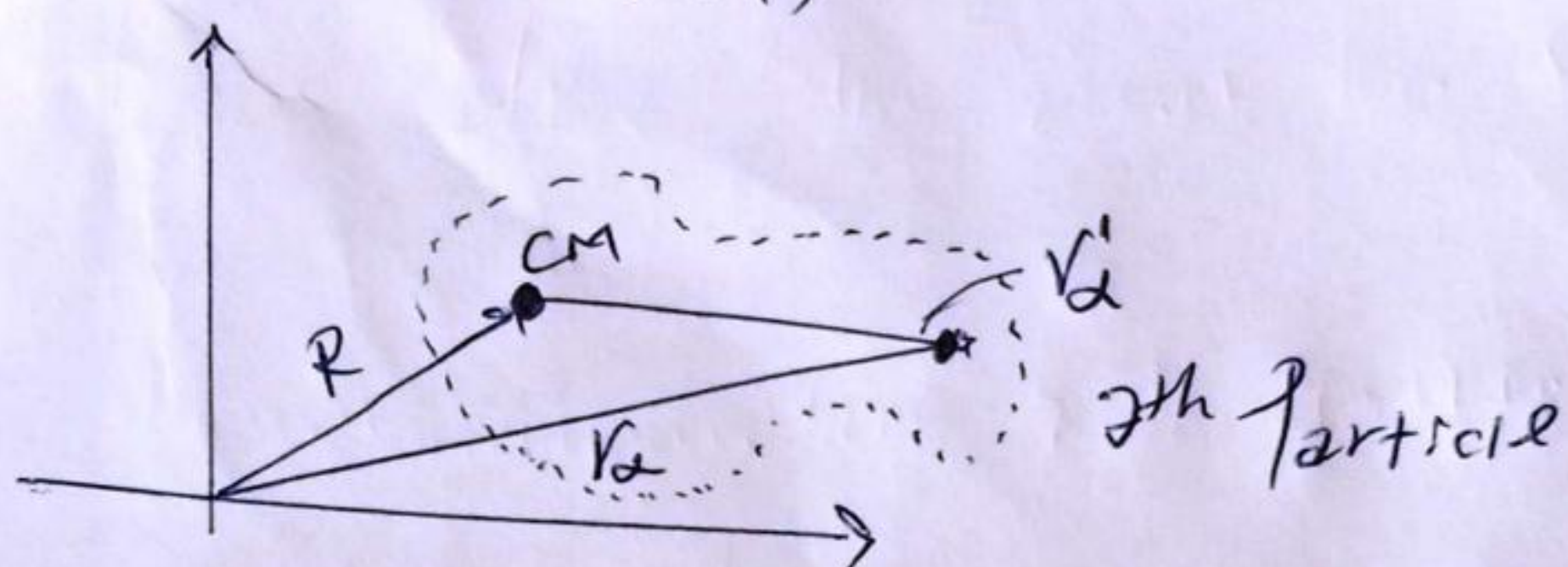
Conservation of Angular Momentum

by Perkeby - 8

Let us describe a system by a position vector with respect to the C.M. to be

$$\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

where $\mathbf{r}'_\alpha$ is the position vector of the particle with respect to the center of mass (C.M.)



The angular momentum of the 2th particle about origin is given by

$$\mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

Summing this expression over α , we have

$$\mathbf{L} = \sum_\alpha \mathbf{L}_\alpha = \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \sum_\alpha (\mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha)$$

$$= \sum_\alpha (\mathbf{r}'_\alpha + \mathbf{R}) \times m_\alpha (\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}})$$

$$= \sum_\alpha m_\alpha \left[(\mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha) + (\mathbf{r}'_\alpha \times \dot{\mathbf{R}}) + (\mathbf{R} \times \dot{\mathbf{r}}'_\alpha) + (\mathbf{R} \times \dot{\mathbf{R}}) \right]$$

The middle two terms can be written as

$$\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha \right) \times \dot{\mathbf{R}} + \mathbf{R} \times \frac{d}{dt} \left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha \right)$$

which vanishes because

$$\sum_{\alpha} m_{\alpha} \dot{r}'_{\alpha} = \sum_{\alpha} m_{\alpha} (\dot{r}_{\alpha} - \dot{R}) = \sum_{\alpha} m_{\alpha} \dot{r}_{\alpha} - \dot{R} \sum_{\alpha} m_{\alpha}$$

$$\sum_{\alpha} m_{\alpha} \dot{r}'_{\alpha} = M \dot{R} - M \dot{R} = 0$$

This indicates that $\sum_{\alpha} m_{\alpha} \dot{r}'_{\alpha}$ specifies the position of the center of mass in the center-of-mass coordinate system and is therefore a null vector. So

$$L = M R \times \dot{R} + \sum_{\alpha} r'_{\alpha} \times p'_{\alpha} = R \times P + \sum_{\alpha} r'_{\alpha} \times p'_{\alpha}$$

this means

The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.

* The time derivative of the angular momentum of the α th particle is

$$\dot{L}_{\alpha} = r_{\alpha} \times \dot{p}_{\alpha} \text{ and}$$

Using conservation of linear momentum

$$\dot{L}_{\alpha} = r_{\alpha} \times \left(F_{\alpha}^{(e)} + \sum_B f_{\alpha B} \right)$$

Summing this expression over α , we have

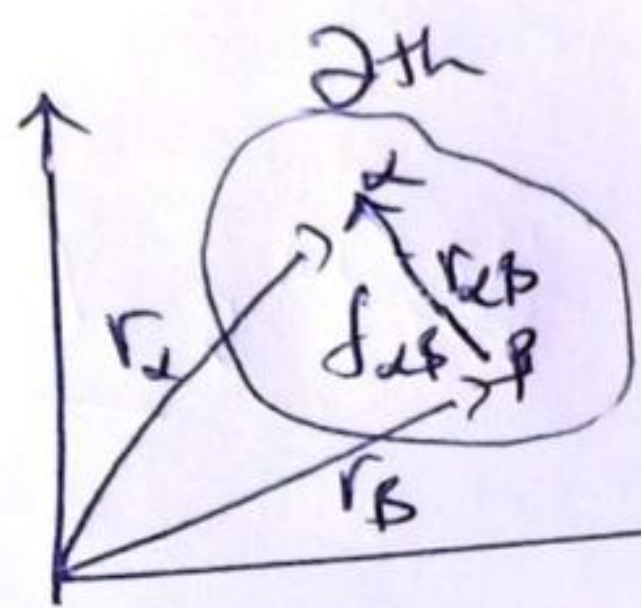
$$\dot{L} = \sum_{\alpha} \dot{L}_{\alpha} = \sum_{\alpha} (r_{\alpha} \times F_{\alpha}^{(e)}) + \sum_{\alpha, B \neq \alpha} (r_{\alpha} \times f_{\alpha B})$$

and The last term may be written as

$$\sum_{\alpha, B \neq \alpha} (r_{\alpha} \times f_{\alpha B}) = \sum_{\alpha < B} [(r_{\alpha} \times f_{\alpha B}) + (r_B \times f_{B\alpha})]$$

The Vector connecting the α th and β th particles can be defined as

$$\mathbf{r}_{\alpha\beta} = \mathbf{r}_\alpha - \mathbf{r}_\beta$$



and then because $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$ we

have

$$\begin{aligned} \sum_{\alpha, \beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}) &= \sum_{\alpha, \beta} (\mathbf{r}_\alpha - \mathbf{r}_\beta) \times \mathbf{f}_{\alpha\beta} \\ &= \sum_{\alpha, \beta} (\mathbf{r}_{\beta\alpha} \times \mathbf{f}_{\alpha\beta}) \end{aligned}$$

Hence, $\mathbf{f}_{\alpha\beta}$ is along the same direction as $\pm \mathbf{r}_{\alpha\beta}$ and

$$\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta} = 0 \text{ and}$$

$$\dot{\mathbf{L}} = \sum_{\alpha} [\mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)}]$$

The right-hand side of this expression is just the sum of all the external torques.

$$\dot{\mathbf{L}} = \sum_{\alpha} \mathbf{M}_\alpha^{(e)} = \mathbf{M}^{(e)} \quad \text{this means}$$

If the net resultant external torques about a given axis vanish, then the total angular momentum of the system about that axis remains constant in time.

The total internal torque must vanish if the internal forces are Central - that is if

$\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$, and the angular momentum of an isolated system cannot be altered without the application of external forces.

Conservation of Energy

by Morkobu. B

- Consider the work done on the system in moving it from a configuration in which all the coordinates r_α are specified to a configuration 2 , in which the coordinates r_α have some different specification. The work done can be expressed as

$$W_R = \int_1^2 F_\alpha dr_\alpha$$

where F_α is the net resultant force acting on a particle α . Using a procedure similar to that used to obtain the relation ship between work done and kinetic energy.

$$W_R = \sum_\alpha \int_1^2 d\left(\frac{1}{2} m_\alpha v_\alpha^2\right) = T_2 - T_1$$

$$\text{where } T = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$$

Using the relation

$$\dot{r}_\alpha = \dot{r}'_\alpha + \dot{R}$$

$$\begin{aligned} v_\alpha^2 &= \dot{r}_\alpha \cdot \dot{r}_\alpha = (\dot{r}'_\alpha + \dot{R}) \cdot (\dot{r}'_\alpha + \dot{R}) \\ &= (\dot{r}'_\alpha \cdot \dot{r}'_\alpha) + 2(\dot{r}'_\alpha \cdot \dot{R}) + (\dot{R} \cdot \dot{R}) \\ &= v'^2_\alpha + 2(\dot{r}'_\alpha \cdot \dot{R}) + v^2 \end{aligned}$$

where $v' = \dot{r}'$ and where v is the velocity of the center of mass.

then

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v'^2_\alpha + \sum_\alpha \frac{1}{2} m_\alpha v^2 + \dot{R} \cdot \frac{d}{dt} \sum_\alpha m_\alpha r'_\alpha$$

by Herkebu. B

from previous argument $\sum_{\alpha} m_{\alpha} \dot{r}_{\alpha} = 0$, and the last term vanishes. Thus,

$$T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{r}_{\alpha}^2 + \frac{1}{2} M \dot{V}^2 \quad \text{which can be stated}$$

- ✓ The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with a velocity of the center of mass and the kinetic energy of motion of the individual particle relative to the center of mass.
- ✓ To obtain the relation b/w work done and potential energy

The work done

$$W_R = \sum_{\alpha} \int_{\alpha} F_{\alpha} dr_{\alpha}$$

$$F_{\alpha} = F_{\alpha}^e + f_{\alpha}$$

$$W_R = \sum_{\alpha} \int_{\alpha} F_{\alpha}^e dr_{\alpha} + \sum_{\alpha, \beta \neq \alpha} \int_{\alpha} f_{\alpha\beta} dr_{\alpha}$$

If the net F_{α}^e and $f_{\alpha\beta}$ are conservative, then they can be expressed from potential energy

$$F_{\alpha}^e = -\nabla_{\alpha} U_{\alpha}$$

$$f_{\alpha\beta} = -\nabla_{\alpha} \bar{U}_{\alpha\beta}$$

where, U_{α} & $\bar{U}_{\alpha\beta}$ are the potential functions but which do not necessarily have the same form.

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} = \frac{\partial}{\partial r} \hat{r}$$

so

The first term becomes

$$\sum_{\alpha} \int_1^2 F_{\alpha} dr_{\alpha} = - \sum_{\alpha} \int_1^2 (\nabla_{\alpha} U_{\alpha}) \cdot dr_{\alpha} = - \sum_{\alpha} U_{\alpha} \int_1^2$$

Also for the second term

$$\begin{aligned} \sum_{\alpha, \beta \neq \alpha} \int_1^2 f_{\alpha\beta} \cdot dr_{\alpha} &= \sum_{\alpha < \beta} \int_1^2 (f_{\alpha\beta} \cdot dr_{\alpha} + f_{\beta\alpha} \cdot dr_{\beta}) \\ &= \sum_{\alpha < \beta} \int_1^2 f_{\alpha\beta} \cdot (dr_{\alpha} - dr_{\beta}) = \sum_{\alpha < \beta} \int_1^2 f_{\alpha\beta} \cdot dr_{\alpha\beta} \end{aligned}$$

where, $dr_{\alpha\beta} = dr_{\alpha} - dr_{\beta}$

* $\bar{U}_{\alpha\beta}$ is function only of the distance between m_{α} and m_{β}

$$d\bar{U}_{\alpha\beta} = (\nabla_{\alpha} \bar{U}_{\alpha\beta}) dr_{\alpha} + (\nabla_{\beta} \bar{U}_{\alpha\beta}) dr_{\beta},$$

$$-\nabla_{\alpha} \bar{U}_{\alpha\beta} = -f_{\alpha\beta}$$

$$-\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha} \quad \text{JD}$$

$$\nabla_{\beta} \bar{U}_{\alpha\beta} = \nabla_{\beta} \bar{U}_{\beta\alpha} = -f_{\beta\alpha} = f_{\alpha\beta}$$

$$\begin{aligned} \text{therefore } d\bar{U}_{\alpha\beta} &= -f_{\alpha\beta} (dr_{\alpha} - dr_{\beta}) \\ &= -f_{\alpha\beta} dr_{\alpha\beta} \end{aligned}$$

Using this results and applying into the second term

$$\sum_{\alpha, \beta \neq \alpha} \int_1^2 f_{\alpha\beta} dr_{\alpha\beta} = - \sum_{\alpha < \beta} \int_1^2 d\bar{U}_{\alpha\beta} = - \sum_{\alpha < \beta} \bar{U}_{\alpha\beta} \int_1^2$$

Combining the first and the second terms

$$W_{12} = - \sum_a U_a \int_1^2 = \sum_{a < b} \bar{U}_{ab} \int_1^2$$

The total potential energy (both internal & external for the system) can be written as

$$U = \sum_a U_a + \sum_{a < b} \bar{U}_{ab}$$

Then $W_{12} = -U \int_1^2 = U_1 - U_2$

Combining this result with kinetic energy

$$T_2 - T_1 = U_1 - U_2 \text{ or}$$

$$T_1 + U_1 = T_2 + U_2$$

so that $E_1 = E_2$

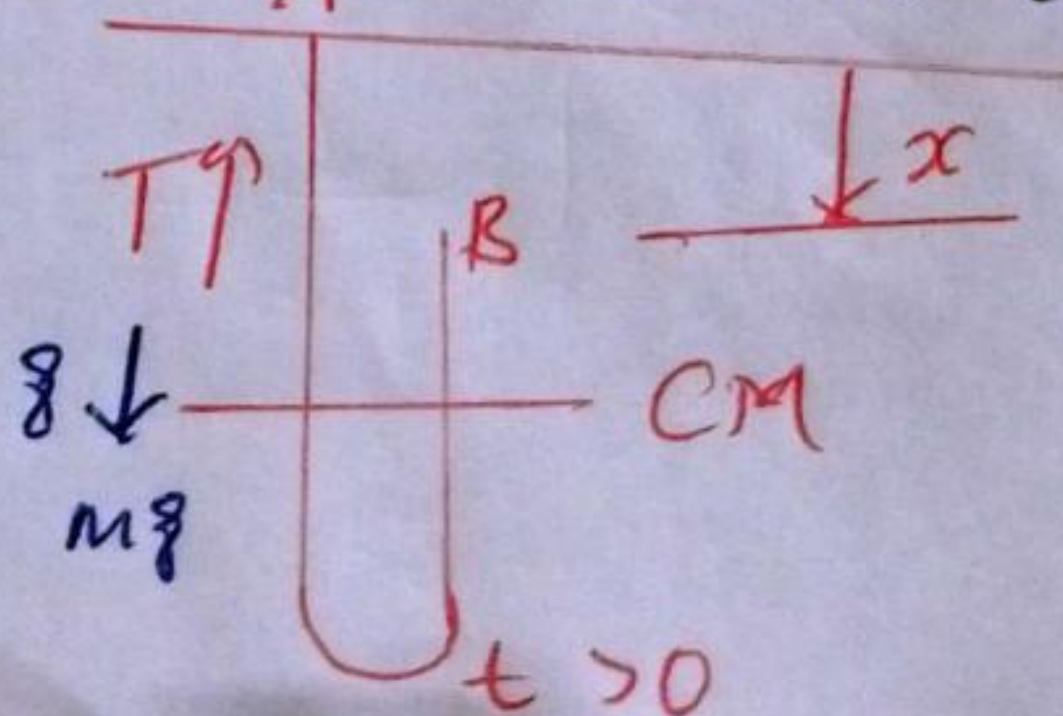
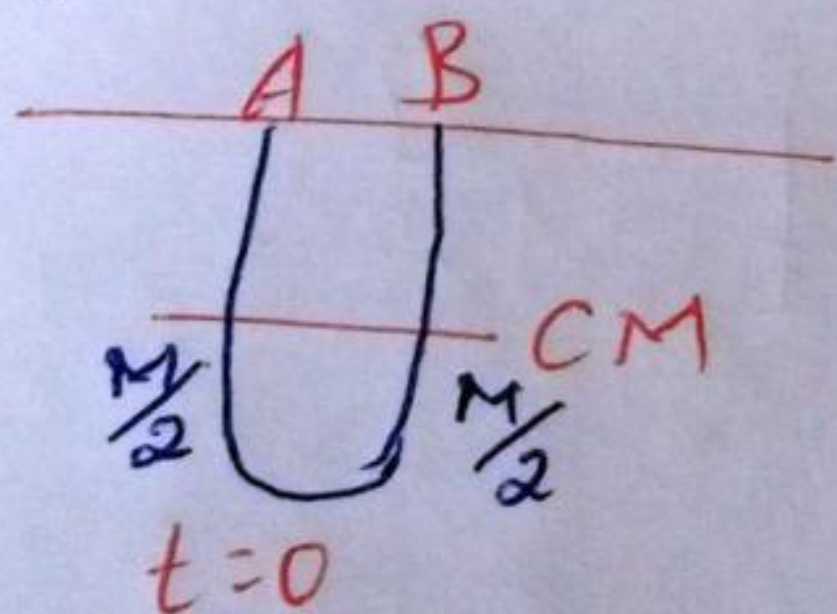
we can say that the total energy for conservative force system is constant.

Exercise

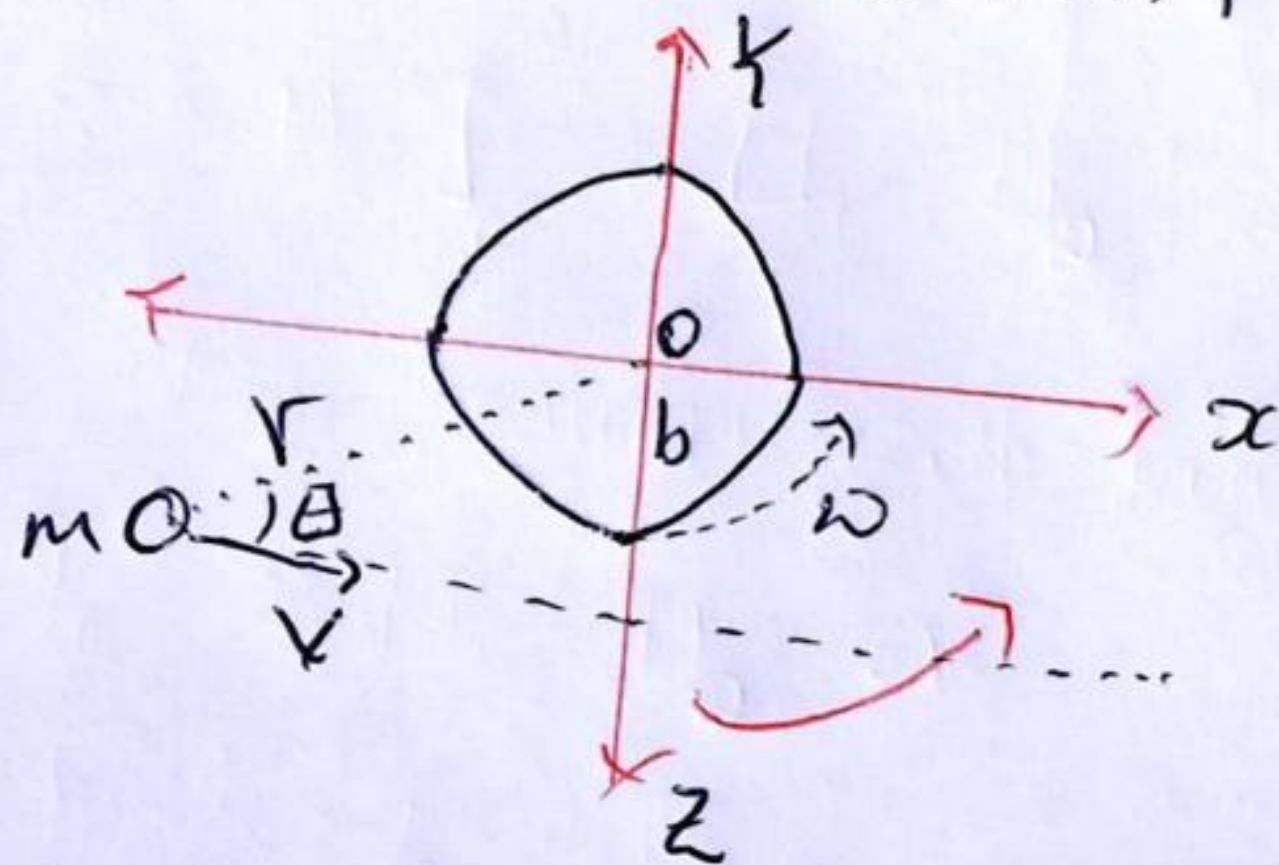
1. A chain of uniform linear mass density (ρ) length l and mass M hangs as shown in figure below. At time $t=0$ the ends A and B are adjacent, but end B is released.

Find the tension in the chain at point A after end B has fallen a distance x by (a) assuming free falling and

(b) by using energy conservation.



- ② A uniform circular turntable (mass M , radius R Center, O) is at rest in the xy plane and is mounted on a frictionless axis which lies along the vertical z -axis through a lump of putty (mass m) with speed v towards the edge of the turn table; so it approaches along a line that passes within a distance $b \neq 0$, as shown in figure below. When the putty hits the turn table, it sticks to the edge, and the two rotate together with angular velocity ω . Find ω .



- ③ A projectile of mass M explodes while in flight into three fragments one mass ($m_1 = M/2$) travels in the original direction of the projectile, (mass $m_2 = M/6$) travels in the opposite direction and mass ($m_3 = M/3$) comes to rest. The energy E released in the explosion is equal to five times the the projectile's kinetic energy explosion. What are the velocities?

